Carlson's generation theorem and reduction to elementary abelian p-groups

Lecture notes for talk #13 of the summer school "Support varieties in modular representation theory"

Sebastian Klein

22/06/2015

1 Introduction

Let G denote a finite group, k a field such that $\mathrm{char}(k) = p$ divides $\#G$, and $\mathbf{D}^{\mathbf{f}}(kG)$ the bounded derived category of finitely generated kG (left-)modules.

1.1 Notation. For an object A of a triangulated category $\mathscr{C}$, we denote by $\langle A\rangle$ the smallest thick subcategory of $\mathscr{C}$ containing A. We denote by $\langle A^\bullet\rangle^\otimes$ the smallest thick $\otimes$-ideal in $\mathbf{D}^{\mathbf{f}}(kG)$ containing a complex $A^\bullet$.

The goal of this talk is to give a proof of the following theorem, which will allow us to deduce the classification result of Benson-Carlson-Rickard later on.

Theorem ("Building", see [3, Theorem 6.6]). *If $M^\bullet, N^\bullet$ are complexes in $\mathbf{D}^{\mathbf{f}}(kG)$ such that $V_G(M^\bullet) \subset V_G(N^\bullet)$, then $N^\bullet$ builds $M^\bullet$, i.e. $M^\bullet \in \langle N^\bullet\rangle^\otimes$.*

Here, $V_G(M^\bullet) \subset \mathrm{Spec}^*(H^*(G,k))$ is the *support variety* of $M^\bullet$ that we have already seen in a previous talk and whose definition will be recalled later on. The proof is based on a result due to Carlson that we discuss next.

2 Carlson's generation theorem

We want to give a proof of the following statement:

2.1 Theorem (see [2]). *Let $E_1,\dots,E_n \subset G$ denote the elementary abelian subgroups of G. Then*

$$\mathbf{D}^{\mathbf{f}}(kG) = \left\langle \bigoplus_{i=1}^{n} k\uparrow_{E_i}^{G} \right\rangle^{\otimes} .$$

2.2 Remark. Carlson's generation theorem is usually stated in a stronger form than above: it asserts that $\mathbf{D}^{\mathbf{f}}(kG) = \langle \bigoplus_{i=1}^n k\uparrow_{E_i}^G \rangle$. However, for our purposes, the version as in Theorem 2.1 suffices (and is easier to prove).

We start with some preparatory results that concern the formal properties of the operations $\langle - \rangle$ and $\langle - \rangle^\otimes$.

2.3 Lemma.

1. *Let $F : \mathscr{C} \to \mathscr{D}$ be an exact functor of triangulated categories and let a, b be objects of $\mathscr{C}$. If $a \in \langle b \rangle$, then $F(a) \in \langle F(b) \rangle$.*
2. *Let $A^\bullet, B^\bullet, C^\bullet \in \mathbf{D}^{\mathrm{f}}(kG)$. If $A^\bullet \in \langle B^\bullet \rangle^\otimes$, then $A^\bullet \otimes C^\bullet \in \langle B^\bullet \otimes C^\bullet \rangle^\otimes$.*
3. *Let*
$$C^\bullet = \cdots \to 0 \to C_1 \to C_2 \to \cdots \to C_n \to 0 \to \cdots$$
be a complex in $\mathbf{D}^{\mathrm{f}}(kG)$. Then $C^\bullet \in \langle \bigoplus_{i=1}^n C_i \rangle$.
4. *Let $H \subset G$ be a subgroup and $A^\bullet, B^\bullet \in \mathbf{D}^{\mathrm{f}}(kH)$. If $A^\bullet \in \langle B^\bullet \rangle^\otimes$, then $A^\bullet\!\uparrow_H^G \in \langle B^\bullet\!\uparrow_H^G \rangle^\otimes$.*

Proof. The first and second statement follow in a straightforward manner from the fact that an exact functor preserves distinguished triangles and direct summands, the third statement is proved by induction on n, progressively truncating the complex $C^\bullet$ and using that short exact sequences of chain complexes give rise to distinguished triangles in the derived category. The proof of the last statement is a little more involved: let us show that the full subcategory

$$\mathscr{S} := \{S^\bullet : S^\bullet\!\uparrow_H^G \in \langle B^\bullet\!\uparrow_H^G \rangle\}$$

is a thick $\otimes$-ideal. Thickness follows easily as $\uparrow_H^G$ is an exact functor and as such preserves distinguished triangles and direct summands. Let us show that it is a $\otimes$-ideal. Let $S^\bullet, T^\bullet \in \mathbf{D}^{\mathrm{f}}(kH)$ and assume $S^\bullet \in \mathscr{S}$. Then we have that

$$(S^\bullet \otimes T^\bullet\!\uparrow_H^G\downarrow_H^G)\!\uparrow_H^G \cong S^\bullet\!\uparrow_H^G \otimes T^\bullet\!\uparrow_H^G \in \langle B^\bullet\!\uparrow_H^G \rangle^\otimes$$

where we used Frobenius reciprocity and that $S^\bullet\!\uparrow_H^G \in \langle B^\bullet\!\uparrow_H^G \rangle^\otimes$. Next, observe that

$$T^\bullet\!\uparrow_H^G\downarrow_H^G = {}_{kH}(kG \otimes_{kH} T^\bullet) = \bigoplus_{[G:H]} T^\bullet$$

as kG is a free kH-module of rank $[G : H]$. It follows that $S^\bullet \otimes T^\bullet$ is a direct summand of $S^\bullet \otimes T^\bullet\!\uparrow_H^G\downarrow_H^G$, which is in $\mathscr{S}$ by the above computation. As $\mathscr{S}$ was a thick subcategory, it follows that $S^\bullet \otimes T^\bullet \in \mathscr{S}$, which shows that $\mathscr{S}$ is a $\otimes$-ideal. As we have $B^\bullet \in \mathscr{S}$ we therefore must have $\langle B^\bullet \rangle^\otimes \subseteq \mathscr{S}$ which implies $A^\bullet \in \mathscr{S}$ by assumption. By definition of $\mathscr{S}$ it follows that $A^\bullet\!\uparrow_H^G \in \langle B^\bullet\!\uparrow_H^G \rangle^\otimes$ which proves the lemma. □

The proof of Theorem 2.1 is now performed in three steps:

1. Reduce to the case where G is a p-group.
2. Use Serre's theorem "on the vanishing of a product of Bocksteins" and a little homological algebra.
3. Induction on $\#G = p^n$.

Let us show how to reduce to the case where G is a p-group.

2.4 Lemma. *Let $S \subset G$ denote a Sylow p-subgroup. The trivial kG-module k is a direct summand of $k\!\uparrow_S^G$.*

Proof. Recall that $k_S\uparrow_S^G = kG \otimes_{kS} k_S$ and we have a surjective augmentation map $\varphi : k_S\uparrow_S^G \to k$ with the property $\varphi(g \otimes a) = a$ for all $a \in k$ and $g \in G$. The submodule of $k_S\uparrow_S^G$ generated by the element $\ell = \sum_{G/S} g \otimes 1$, the sum of a complete set of left coset representatives of S in G, is G-invariant. Furthermore, $\varphi(\ell) = [G : S] \mod p$, which is non-zero in k. Hence, the map

$$\begin{aligned} \psi : k &\to k_S\uparrow_S^G \\ 1 &\mapsto \frac{1}{[G:S]} \cdot \ell \end{aligned}$$

is a kG-homomorphism that is a right-inverse for φ. □

2.5 Proposition. *Assume Theorem 2.1 holds for all p-groups G. Then it holds for all G.*

Proof. With the assumptions as in Theorem 2.1 and G not a p-group, let $S \subset G$ be a Sylow p-subgroup. By assumption, we have

$$\mathbf{D}^{\mathbf{f}}(kS) = \left\langle \bigoplus_{i=1}^{n} k\uparrow_{E_i}^S \right\rangle^{\otimes}$$

where $E_i, i = 1, \ldots, n$ are the elementary abelian subgroups of S. In particular, we have that $k \in \langle \bigoplus_{i=1}^n k\uparrow_{E_i}^S \rangle^{\otimes}$ which implies by Lemma 2.3 that

$$k\uparrow_S^G \in \left\langle \bigoplus_{i=1}^{n} k\uparrow_{E_i}^G \right\rangle^{\otimes} .$$

Using Lemma 2.4, it follows that $k \in \langle \bigoplus_{i=1}^n k\uparrow_{E_i}^G \rangle^{\otimes}$ which implies that $\langle \bigoplus_{i=1}^n k\uparrow_{E_i}^G \rangle^{\otimes} = \mathbf{D}^{\mathbf{f}}(kG)$. □

The crucial step in the proof of Theorem 2.1 is the second one, which we sketch next: let us first review some homological algebra concerning Ext functors. For $A, B \in kG$—mod there are two different perspectives on the group $\mathrm{Ext}_{kG}^n(A, B)$. We may view an element $f \in \mathrm{Ext}_{kG}^n(A, B)$ as an exact sequence

$$f_1 : 0 \to B \xrightarrow{d^1} E_1 \to \cdots \to E_n \xrightarrow{d^{n+1}} A \to 0$$

or alternatively as a morphism

$$f_2 : A \to B[n]$$

in $\mathbf{D}^{\mathbf{f}}(kG)$. Given an exact sequence f_1, the corresponding morphism f_2 is constructed as follows: consider the diagram of chain complexes $A \xleftarrow{s} \tilde{f} \xrightarrow{t} B[n]$ given by

$$\begin{array}{ccccccccccccccccc}
B[n] = & \cdots & \to & 0 & \to & B & \to & 0 & \to & 0 & \to \cdots \to & 0 & \to & 0 & \to & \cdots \\
 & & & & & \uparrow{\scriptstyle \mathrm{id}_B} & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
\tilde{f} = & \cdots & \to & 0 & \to & B & \xrightarrow{d^1} & E_1 & \xrightarrow{d^2} & E_2 & \to \cdots \xrightarrow{d^n} & E_n & \to & 0 & \to & \cdots \\
 & & & & & \downarrow & & \downarrow & & \downarrow & & \downarrow{\scriptstyle d^{n+1}} & & \downarrow & & \\
A = & \cdots & \to & 0 & \to & 0 & \to & 0 & \to & 0 & \to \cdots \to & A & \to & 0 & \to & \cdots
\end{array}$$

We see that s is a quasi-isomorphism and therefore invertible in the derived category. The map f_2 is defined as the composition $s^{-1} \circ t$. It is not hard to check that the mapping cone of f_2 is quasi-isomorphic to the complex

$$\cdots \to 0 \to E_1 \xrightarrow{-d^2} E_2 \to \cdots \xrightarrow{-d^n} E_n \to 0 \to \cdots$$

with E_1 in degree n. Let us summarize the preceding discussion:

2.6 Lemma. *Let $A, B \in kG$—mod and $f \in \mathrm{Ext}^n(A,B)$. Assume that f is represented by an exact sequence*

$$0 \to B \xrightarrow{d^1} E_1 \xrightarrow{d^2} \cdots \xrightarrow{d^n} E_n \xrightarrow{d^{n+1}} A \to 0 .$$

If we identify $\mathrm{Ext}^n(A,B)$ with $\mathrm{Hom}_{\mathbf{D}^{\mathrm{f}}(kG)}(A, B[n])$, then $\mathrm{cone}(f) \in \mathbf{D}^{\mathrm{f}}(kG)$ is quasi-isomorphic to the complex

$$\cdots \to 0 \to E_1 \xrightarrow{-d^2} E_2 \to \cdots \xrightarrow{-d^n} E_n \to 0 \to \cdots$$

with E_1 in degree n.

2.7 Remark. The above operation preserves compositions: if $f \in \mathrm{Ext}^n_{kG}(A,B)$ and $g \in \mathrm{Ext}^m_{kG}(B,C)$, then the construction takes the Yoneda splice of two exact sequences representing f and g to their compositions as morphisms in $\mathbf{D}^{\mathrm{f}}(kG)$.

The next theorem is the key result for the proof Theorem 2.1.

2.8 Theorem (Serre)**.** *Let G be a p-group which is not elementary abelian and let $H_1, \ldots, H_m$ be the maximal subgroups of G. Then there exist elements $\beta(z_i) \in \mathrm{H}^2(G,k) = \mathrm{Ext}^2(k,k), i = 1, \ldots, m$, represented by exact sequences of the form*

$$0 \to k_G \to k_{H_i}{\uparrow}^G_{H_i} \xrightarrow{\alpha_i} k_{H_i}{\uparrow}^G_{H_i} \to k_G \to 0 ,$$

such that their product $\beta(z_1) \cdot \ldots \cdot \beta(z_m)$ vanishes in $\mathrm{H}^{2m}(G,k) = \mathrm{Ext}^{2m}(k,k)$.

2.9 Corollary. *Let G be a p-group which is not elementary abelian and let $H_1, \ldots, H_m$ be the collection of maximal subgroups of G. Then*

$$\mathbf{D}^{\mathrm{f}}(kG) = \left\langle \bigoplus_{i=1}^m k_{H_i}{\uparrow}^G_{H_i} \right\rangle^{\otimes}$$

Proof (adapted from [1, Proof of Theorem 4.3]). We let $\beta(z_i) \in \mathrm{H}^2(G,k) = \mathrm{Ext}^2(k,k), i = 1, \ldots, m$ be as in Theorem 2.8. By Lemma 2.6, they correspond to morphisms $\beta(z_i) : k \to k[2]$ in $\mathbf{D}^{\mathrm{f}}(kG)$ with

$$\mathrm{cone}(\beta(z_i)) \cong \cdots \to 0 \to k_{H_i}{\uparrow}^G_{H_i} \xrightarrow{\alpha_i} k_{H_i}{\uparrow}^G_{H_i} \to 0 \to \cdots .$$

From Lemma 2.3, we see that the truncation of the Yoneda splice of all the exact sequences representing $\beta(z_i)$

$$0 \to k_{H_1}{\uparrow}^G_{H_1} \xrightarrow{\alpha_1} k_{H_1}{\uparrow}^G_{H_1} \to k_{H_2}{\uparrow}^G_{H_2} \xrightarrow{\alpha_2} k_{H_2}{\uparrow}^G_{H_2} \to \cdots \to k_{H_m}{\uparrow}^G_{H_m} \xrightarrow{\alpha_m} k_{H_m}{\uparrow}^G_{H_m} \to 0 \qquad (1)$$

is in $\langle \bigoplus_{i=1}^m k_{H_i}{\uparrow}^G_{H_i} \rangle$. By Remark 2.7, the complex (1) is the mapping cone of $\beta(z_1) \cdot \ldots \cdot \beta(z_m) \in \mathrm{Hom}_{\mathbf{D}^{\mathrm{f}}(kG)}(k, k[2m])$. But by Serre's theorem $\beta(z_1) \cdot \ldots \cdot \beta(z_m) = 0$, so its mapping cone (1) is isomorphic to $k[1] \oplus k[2m]$. Thus, $k \in \langle \bigoplus_{i=1}^m k_{H_i}{\uparrow}^G_{H_i} \rangle$, which implies that $\langle \bigoplus_{i=1}^m k_{H_i}{\uparrow}^G_{H_i} \rangle^{\otimes} = \mathbf{D}^{\mathrm{f}}(kG)$. $\square$

We are now ready for the last step of the proof of the generation theorem.

Proof of Theorem 2.1. By Proposition 2.5, we may assume that G is a p-group of order p^n, and the proof will proceed by induction on n. For $n = 1$, the statement is trivial as G is elementary abelian itself and

$$\mathbf{D}^{\mathbf{f}}(kG) = \langle k \rangle^{\otimes} = \langle k{\uparrow}_G^G \rangle^{\otimes} .$$

Therefore, assume we have proved the theorem for all $n < n_0$ and G has order n_0. If G is elementary abelian, we are done again, so assume it's not. By Corollary 2.9, we have

$$\mathbf{D}^{\mathbf{f}}(kG) = \left\langle \bigoplus_{i=1}^{m} k_{H_i}{\uparrow}_{H_i}^{G} \right\rangle^{\otimes}$$

with $H_1, \ldots, H_m$ be the maximal subgroups of G. All the H_i are p-groups of order $< p^{n_0}$ and so by the induction hypothesis, we have for all i

$$\mathbf{D}^{\mathbf{f}}(kH_i) = \left\langle \bigoplus_{s=1}^{n_i} k{\uparrow}_{E_{i_s}}^{H_i} \right\rangle^{\otimes}$$

with $E_{i_1}, \ldots, E_{i_{n_i}}$ the elementary abelian subgroups of H_i. Now notice that for each i, we have $k \in \langle \bigoplus_{s=j}^{n_i} k{\uparrow}_{E_{i_s}}^{H_i} \rangle^{\otimes}$, and by Lemma 2.3, we have that $k{\uparrow}_{H_i}^{G} \in \langle \bigoplus_{s=j}^{n_i} k{\uparrow}_{E_{i_s}}^{G} \rangle^{\otimes}$. It follows that

$$\bigoplus_{i=1}^{m} k_{H_i}{\uparrow}_{H_i}^{G} \in \left\langle \bigoplus_{s,i} k{\uparrow}_{E_{i_s}}^{G} \right\rangle^{\otimes}$$

from which we deduce the inclusion

$$\mathbf{D}^{\mathbf{f}}(kG) = \left\langle \bigoplus_{i=1}^{m} k_{H_i}{\uparrow}_{H_i}^{G} \right\rangle^{\otimes} \subseteq \left\langle \bigoplus_{s,i} k{\uparrow}_{E_{i_s}}^{G} \right\rangle^{\otimes}$$

that must then be an equality. □

3 Proof of the main theorem

We will now recall a notion of support for complexes $C^\bullet \in \mathbf{D}^{\mathbf{f}}(kG)$ that we already saw in a previous talk. Let us define

$$\mathrm{H}^\bullet(G,k) = \begin{cases} \bigoplus_i \mathrm{Ext}^i_{kG}(k,k) & \text{if } p = 2 \\ \bigoplus_i \mathrm{Ext}^{2i}_{kG}(k,k) & \text{otherwise} \end{cases}$$

In both cases, $\mathrm{H}^\bullet(G,k)$ is a commutative noetherian k-algebra and for any complex $C^\bullet$ of kG-modules, there is a k-algebra homomorphism

$$- \otimes C^\bullet : \mathrm{H}^*(G,k) \to \mathrm{Ext}^*(C^\bullet, C^\bullet)$$

which gives $\mathrm{Ext}^*(C^\bullet, C^\bullet)$ the structure of a module over $\mathrm{H}^\bullet(G,k)$. It is a finitely generated module when $C^\bullet$ is in $\mathbf{D}^{\mathbf{f}}(kG)$ by the Evens-Venkov theorem.

3.1 Definition. Let $C^\bullet$ be a complex in $\mathbf{D}^{\mathbf{f}}(kG)$. Its *support* is the subset

$$V_G(C^\bullet) := \mathrm{Supp}_{\mathrm{H}^\bullet(G,k)}\mathrm{Ext}^*(C^\bullet, C^\bullet) \subset \mathrm{Spec}^*(\mathrm{H}^\bullet(G,k)) .$$

We now relate this notion of support to the one discussed in the previous lectures. Let E be an elementary abelian p-group of rank r. Its group algebra is isomorphic to the artinian complete intersection $R := k[z_1, \dots, z_r]/(z_1^p, \dots, z_r^p)$. In a previous talk, we defined for any complex $C^\bullet$ in $\mathbf{D}^{\mathbf{f}}(R)$ a support variety $V_R(C^\bullet)$ using the action of $\mathrm{Ext}_R^*(k,k)$ on the module $\mathrm{Ext}^*(k, C^\bullet)$ (restricted to a polynomial subring $k[\boldsymbol{\theta}]$). It turns out that the two support varieties $V_R(C^\bullet)$ and $V_E(C^\bullet)$ coincide.

3.2 Lemma. *E be an elementary abelian p-group of rank r with group algebra isomorphic to $R := k[z_1, \dots, z_r]/(z_1^p, \dots, z_r^p)$. For any complex $C^\bullet$ in $\mathbf{D}^{\mathbf{f}}(kE)$, we have an equality*

$$V_E(C^\bullet) = V_R(C^\bullet) .$$

Idea of the proof. One first shows that for any homogeneous prime ideal $\mathfrak{p} \in \mathrm{Spec}^*(\mathrm{H}^\bullet(G,k))$, we have

$$\mathrm{Ext}^*(C^\bullet, C^\bullet)_{\mathfrak{p}} = 0 \Leftrightarrow \mathrm{Ext}^*(B^\bullet, C^\bullet)_{\mathfrak{p}} \forall B^\bullet \in \mathbf{D}^{\mathbf{f}}(kE) \Leftrightarrow \mathrm{Ext}^*(k^\bullet, C^\bullet)_{\mathfrak{p}} = 0$$

where the last step follows from $\mathbf{D}^{\mathbf{f}}(kE) = \langle k \rangle$. Since the inclusion of $k[\boldsymbol{\theta}]$ into $\mathrm{Ext}^*(k,k)$ factors through $\mathrm{H}^\bullet(G,k)$ and becomes an isomorphism after killing all nilpotents (which don't contribute to the support anyway), the equality of support varieties follows. □

Before we come to the proof of the main theorem, recall from a previous lecture that the restriction functor $\downarrow_E^G$: kG–mod → kE–mod induces a map of graded rings $H^\bullet(G,k) \to H^\bullet(E,k)$ which in turn induces a map of support varieties

$$\mathrm{res}^* : \mathrm{Spec}^*(H^*(E,k)) \to \mathrm{Spec}^*(H^*(G,k)) .$$

3.3 Lemma. *Let $C^\bullet$ be a complex in $\mathbf{D}^{\mathbf{f}}(kG)$. Then*

$$V_E\left(C^\bullet\downarrow_E^G\right) = (\mathrm{res}^*)^{-1}\left(V_G(C^\bullet)\right) .$$

As mentioned before, the proof of Lemma 3.3 is not so easy and can, for example, be deduced from the Quillen stratification theorem.

We have now assembled all of the tools to give a short proof of the main theorem of this lecture.

3.4 Theorem. *If $M^\bullet, N^\bullet$ are complexes in $\mathbf{D}^{\mathbf{f}}(kG)$ such that $V_G(M^\bullet) \subset V_G(N^\bullet)$, then $M^\bullet \in \mathrm{Thick}^\otimes(N^\bullet)$.*

Proof. Let E be an elementary abelian subgroup of G. Then by Lemma 3.3 and our assumption we obtain

$$V_E(M^\bullet\downarrow_E^G) = (\mathrm{res}^*)^{-1}\left(V_G(M^\bullet)\right) \subset (\mathrm{res}^*)^{-1}\left(V_G(N^\bullet)\right) = V_E(N^\bullet\downarrow_E^G)$$

and by Lemma 3.2 and a result of a previous lecture, this implies that $M^\bullet\downarrow_E^G \in \langle N^\bullet\downarrow_E^G\rangle$. By Lemma 2.3 and using Frobenius reciprocity it follows that

$$M^\bullet\downarrow_E^G\uparrow_E^G = M^\bullet \otimes_k k\uparrow_E^G \in \langle N^\bullet\downarrow_E^G\uparrow_E^G\rangle = \langle N^\bullet \otimes_k k\uparrow_E^G\rangle \subset \langle N^\bullet\rangle^\otimes .$$

Next, let $E_1, \dots, E_t$ denote the elementary abelian p-subgroups of G. By Theorem 2.1, we have that $k \in \langle \bigoplus_i k_{E_i}\uparrow_{E_i}^G\rangle^\otimes$ and from Lemma 2.3 we obtain that $M^\bullet \in \langle \bigoplus_i M^\bullet \otimes_k k\uparrow_{E_i}^G\rangle^\otimes$. By our previous calculation, $M^\bullet \otimes_k k\uparrow_{E_i}^G$ is contained in $\langle N^\bullet\rangle^\otimes$ for $i = 1, \cdots, t$, and so we must have that $\bigoplus_i M^\bullet \otimes_k k\uparrow_{E_i}^G \in \langle N^\bullet\rangle^\otimes$ and thus

$$M^\bullet \in \left\langle \bigoplus_i M^\bullet \otimes_k k\uparrow_{E_i}^G \right\rangle^\otimes \subseteq \langle N^\bullet\rangle^\otimes$$

which finishes the proof of the theorem. □

References

[1] Paul Balmer. *Separable extensions in tt-geometry and generalized Quillen stratification*. 2013. eprint: `arXiv:1309.1808`.

[2] Jon F. Carlson. “Cohomology and induction from elementary Abelian subgroups.” In: *Q. J. Math.* 51.2 (2000), pp. 169–181. ISSN: 0033-5606; 1464-3847/e.

[3] Jon F. Carlson and Srikanth B. Iyengar. “Thick subcategories of the bounded derived category of a finite group.” In: *Trans. Am. Math. Soc.* 367.4 (2015), pp. 2703–2717. ISSN: 0002-9947; 1088-6850/e.